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On strong integrality properties of the perfect matching polytope

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Abstract This paper investigates integrality properties of perfect matching polytopes, focusing on box-total dual integrality and integer decomposition properties.

We begin by characterizing the graphs whose perfect matching polytope is a slice of the nonnegative orthant, identifying these as the solid graphs introduced by de Carvalho, Lucchesi, and Murty in "On a Conjecture of Lovász Concerning Bricks: I. The Characteristic of a Matching Covered Graph" (*Journal of Combinatorial Theory, Series B)*.

As a result, we show that the perfect matching polytope of solid graphs admits a compact description, and we establish that deciding the box-total dual integrality of a perfect matching polytope can be done in polynomial time.

Additionally, we characterize the conditions under which perfect matching polytopes of two fundamental graph classes, namely near-bricks and bicritical graphs, are box-totally dual integral. We discuss implications of these results for identifying perfect matching polytopes with the integer decomposition property. This in particular unveils a new positive case of the generalized Berge-Fulkeron conjecture.

Keywords Perfect matching polytope · Box-totally dual integral polyhedron · Integer decomposition property.

1 Introduction

This paper investigates integrality properties of perfect matching polytopes, widely studied in combinatorial optimization, with a focus on box-total dual integrality and integer decomposition properties, two fundamental notions in integer programming.

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Totally dual integral and box-totally dual integral systems were introduced as a versatile framework for establishing various min-max relations in combinatorial optimization [39]. A rational linear system $Ax \leqslant b$ is *totally dual integral (TDI)* if the minimum in the linear duality equation

$$\max\{w^\top x \colon Ax \leqslant b, x \geqslant \mathbf{0}\} = \min\{b^\top y \colon A^\top y \geqslant w, y \geqslant \mathbf{0}\}$$

has an integer optimal solution for every integer vector w such that the optimum is finite. If a system is TDI, then the right-hand side can be chosen integer if and only if the polyhedron described by the system is integer [24].

A stronger property for a system $Ax \leqslant b$ is the *box-total dual integrality (box-TDI)*, which holds when $Ax \leqslant b, \ell \leqslant x \leqslant u$ is TDI for all rational vectors ℓ and u (with possible infinite components). Classical examples of box-TDI systems are those defined by *totally unimodular* matrices, which are the matrices whose minors are all $0, \pm 1$ [38]. For instance, Kőnig's Theorem [32] and the MaxFlow-MinCut Theorem of Ford and Fulkerson [25] can be deduced from the total unimodularity of incidence matrices of bipartite graphs and of directed graphs, respectively. While every polyhedron can be described by a TDI system [27], there are polyhedra that are not described by any box-TDI system. A polyhedron is *box-TDI* if it can be described by a box-TDI system. Cook [12] proved that any TDI system describing a box-TDI polyhedron is a box-TDI system.

Box-TDI systems and polyhedra have been actively studied the past three decades. Box-Mengerian matroid ports are characterized in [7]. Series-parallel graphs form a class in which several polyhedra turn out to be box-TDI: a box-TDI system describes their 2-edge-connected spanning subgraph polyhedron [8] and is generalized for k-edge-connectivity [1]; [14] provides several other box-TDI systems; [2] proves the box-TDIness of their flow cone. Regarding box-perfect graphs, which are the perfect graphs having a box-TDI stable set polytope, new classes of box-perfect graphs are introduced in [21], and a weak box-perfect graph theorem is given in [9]. Matricial and geometrical characterizations of box-TDI polyhedra can be found in [11]. Complexity results regarding box-TDIness are given in [10]. Graphs for which the matching polytope is box-TDI are characterized in [20].

A polyhedron P has the *integer decomposition property* if every integer point in the k-th dilation kP of P is the sum of k integer points from P, for all $k \in \mathbb{Z}_{>0}$. If P has the integer decomposition property, then P is integer, and every face of P also has the integer decomposition property [38, Section 22.10]. Originally introduced in integer programming by Baum and Trotter [3], the integer decomposition property has since been studied in fields such as algebraic geometry and combinatorial commutative algebra [28].

Several classes of polyhedra are known to have the integer decomposition property, including projections of polyhedra defined by totally unimodular matrices [40], polyhedra defined by nearly totally unimodular matrices [26], certain polyhedra defined by k-balanced matrices [45], and stable set polytopes of claw-free t-perfect graphs and h-perfect line-graphs [5]. Additional connections with Fulkerson's theory of blocking and anti-blocking polyhedra are explored in [4]. The matching polytope of a bipartite graph has the integer decomposition property, since it is described by a totally unimodular matrix. This is generalized in [43] to matchings of size $k \leqslant \lfloor n/2 \rfloor$ of a bipartite graph with n vertices.

In a graph, a *matching* is a subset of pairwise nonincident edges, and a *perfect matching* is a matching that covers all the vertices. The *matching polytope* of a graph is the convex hull of the incidence vectors of its matchings. Similarly, the *perfect matching polytope* of a graph is the convex hull of the incidence vectors of its perfect matchings. Since the perfect matching polytope is a face of the matching polytope and as box-TDIness is preserved under taking faces, Ding, Tan, and Zang [20]'s characterization in terms of forbidden subgraphs gives sufficient conditions for the box-TDIness of the perfect matching polytope. However, as the perfect matching polytope of a subgraph needs not to be a face of the perfect matching polytope of the original graph, there is no characterization of its box-TDIness in terms of forbidden subgraphs.

In this paper, we report progress on the box-TDIness and the integer decomposition property of the perfect matching polytope.

Contributions. Our contributions are threefold. First, we characterize the graphs for which the perfect matching polytope is the intersection of its affine hull with the nonnegative orthant: these are precisely the so-called solid graphs. This extends a result of de Carvalho et al. [17].

Second, we characterize the box-TDIness of the perfect matching polytope of two fundamental classes of graphs in the context of perfect matchings: near-bricks and bicritical graphs. This graphic characterization involves odd intercyclicity and follows from the study of the impact of tight cut contractions on the box-TDIness of the perfect matching polytope. More precisely, we prove that contracting a tight cut preserves the box-TDI of the perfect matching polytope. We observe that the converse does not hold in general. Nevertheless, for 2-separation cuts, which are one of the two particular classes of tight cuts in which one can step in during the tight cut decomposition, we prove that the converse holds.

It is known that a box-TDI polyhedron has a box-TDI affine hull. We prove that the converse holds for the perfect matching polytope, that is, the perfect matching polytope is box-TDI if and only if its affine hull is. As a consequence, determining whether the perfect matching polytope is box-TDI can be done in polynomial time. This contrasts with the general problem of deciding whether a polyhedron is box-TDI, which is co-NP-complete [10]. As another consequence, box-TDI perfect matching polytopes have the integer decomposition property, unlike in the general case. In particular, we obtain a new positive case of the famous generalized Berge-Fulkerson conjecture, due to Seymour [42]. We highlight that the converse does not hold by providing a general class of graphs whose perfect matching polytopes have the integer decomposition property but is not box-TDI.

Outline. In Section 2, we recall the results from the literature relevant to our work. In Section 3, we characterize the graphs for which the perfect matching polytope is the intersection of its affine hull with the nonnegative orthant. Moreover, we prove that the box-TDIness of perfect matching polytopes is characterized by that of its affine hull. This yields a polynomial-time algorithm for verifying the box-TDIness of the perfect matching polytope. In Section 4, we characterize which near-bricks and bicritical graphs have a box-TDI perfect matching polytope. In Section 5, we discuss the integer decomposition property of the perfect matching polytope.

2 Preliminaries

In this section, we give the results that we shall use throughout the paper: about box-TDI polyhedra, the perfect matching polytope, matching covered graphs and their tight cut decompositions, and the affine hull of perfect matchings. Finally, we discuss differences between the box-TDIness of the matching polytope and that of the perfect matching polytope.

In this work, all considered graphs are undirected. Without loss of generality, we assume all graphs to be connected with at least one edge, as our results extend immediately to general undirected loopless graphs. A special role is played by *odd intercyclic* graphs, which are the graphs having no two vertex-disjoint odd cycles.

For a given graph $G = (V, E)$ we denote by A_G the (vertex-edge) incidence matrix of G, whose v, e entry is 1 if v is an extremity of e, and 0 otherwise. For $C \subseteq E$, we denote by χ^C the incidence vector of C. Throughout, $\mathbf{0}$ (resp. $\mathbf{1}$) will respectively denote a zero (resp. one) entrywise matrix of appropriate size.

2.1 Box-total dual integrality, equimodularity, and integer decomposition property.

A matrix is *equimodular* if it has full row rank and all maximal nonzero minors are equal up to the sign. Equimodularity is characterized in terms of total unimodularity.

Theorem 1 (Heller [29]) A full row rank $m \times n$ matrix A is equimodular if and only if $B^{-1}A$ is totally unimodular for every nonsingular $m \times m$ submatrix B of A. Equivalently, $B^{-1}A$ is a $\{0, \pm 1\}$-matrix for every nonsingular $m \times m$ submatrix B of A.

It is well-known that the incidence matrix of a graph is totally unimodular if and only if the graph is bipartite [30]. The following result characterizes the class of graphs whose incidence matrix is equimodular.

Theorem 2 (Chervet et al. [10]) The incidence matrix of a connected nonbipartite graph G is equimodular if and only if G is odd intercyclic.

Box-TDI polyhedra are characterized in terms of equimodular matrices as follows. A matrix is *face-defining* for a polyhedron P if it has full row rank and describes the affine hull of some face of P.

Theorem 3 (Chervet et al. [11]) A polyhedron is box-TDI if and only if all its face-defining matrices are equimodular. Equivalently, each of its faces admits an equimodular face-defining matrix.

Theorem 3 contains the well-known fact that polyhedra described by totally unimodular matrices are box-TDI [38, Chapter 22]. Moreover, these polyhedra have the integer decomposition property.

Theorem 4 (Baum and Trotter [3]) If A is totally unimodular and b integer, then $P = \{x\colon Ax \leqslant b\}$ has the integer decomposition property.

2.2 Perfect matchings.

We refer to [34] for an extensive introduction about perfect matchings. We first recall the following well-known characterization of the existence of a perfect matching, where $\mathcal{O}(G)$ denotes the family of connected components of odd cardinality of the graph G.

Theorem 5 (Tutte [44]) A graph $G = (V, E)$ has a perfect matching if and only if $|\mathcal{O}(G \backslash S)| \leqslant |S|$ for all $S \subseteq V$.

When looking at the perfect matchings of a graph, one may restrict not only to connected graphs, but even to *matching covered* graphs, which are the connected graphs in which every edge belongs to a perfect matching. By definition, a matching covered graph is 2-*connected*, that is, no vertex removal disconnects the graph. The following theorem of Lovász [35] characterizes matching covered graphs. A subset S of vertices of a graph G is a *barrier* if $|\mathcal{O}(G \backslash S)| = |S|$.

Theorem 6 (Lovász [35]) A graph having a perfect matching is matching covered if and only if each barrier is composed of pairwise nonadjacent vertices.

Let $G = (V, E)$ be a matching covered graph. For a subset X of vertices, $E(X)$ is the set of edges of G having both extremities in X, and $\delta(X)$ denotes the *cut* determined by X, that is, the set of edges having precisely one extremity in X. The *shores* of a cut $\delta(X)$ are X and $\overline{X} = V\backslash X$. A cut is *trivial* if one of its shores is a singleton. For $X \subseteq V$, *contracting X to a single vertex x* means replacing X by a new vertex x with $\delta(x) = \delta(X)$, and we denote the resulting graph by G/X. The two graphs G/X and $G/\overline{X}$ are referred to as the two $\delta(X)$*-contractions* of G. A cut C of G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. A cut C of G is a *separating cut* if both of the C-contractions of G are also matching covered. Every tight cut of a matching covered graph is separating, but the converse does not hold. For example, in the prism $\overline{C}_6 \coloneqq K_6\backslash E(C_6)$, which is the complement of a cycle of length 6, the set of edges contained in no triangle forms a separating cut, which is not tight since it is also a perfect matching. Two tight cuts $\delta(X)$ and $\delta(Y)$ are *laminar* if X and Y are either disjoint or one of X and Y contains the other one. A *laminar family of tight cuts* is a family of pairwise laminar tight cuts.

A graph is *solid* if it is matching covered and all its separating cuts are tight. A matching covered graph free of nontrivial tight cuts is a *brace* if it is bipartite and a *brick* otherwise. A graph is *bicritical* if removing any couple of vertices yields a graph having a perfect matching. A graph is a brick if and only if it is 3-connected and bicritical [35]. Typical examples of bricks are the complete graph on four vertices K_4, $\overline{C}_6$, and the Petersen graph [35].

For a matching covered graph, the following holds.

Theorem 7 (Edmonds et al. [23]) Let G be a matching covered graph, and $\delta(X)$ and $\delta(Y)$ be two laminar tight cuts of G. If $\delta(Y)$ is a cut of G/X, then $\delta(Y)$ is tight for G/X.

Let $\mathcal{F}$ be a laminar family of nontrivial tight cuts of a matching covered graph G. Note that contracting any shore of a tight cut of $\mathcal{F}$ produces a smaller matching covered graph. By Theorem 7, the cuts from $\mathcal{F}$ that remain nontrivial cuts in the resulting graph remain tight. Repeating this process yields a graph in which all the cuts of $\mathcal{F}$ either vanish or become trivial. We refer to the graphs obtained through this process as $\mathcal{F}$*-contractions* of G. See Figure 1 illustrates the $\delta(X)$-contractions of G when $\mathcal{F} = \{\delta(X)\}$.

When $\mathcal{F}$ is an inclusionwise maximal laminar family of nontrivial tight cuts, the resulting $\mathcal{F}$-contractions contain no nontrivial tight cuts and hence are either bricks or braces. Thus, given such an $\mathcal{F}$, a *tight cut decomposition* of G is the collection of all bricks and braces obtained as $\mathcal{F}$-contractions [35]. A fundamental result by Lovász [35] asserts that any tight cut decomposition of G yields the same set of bricks and braces (up to edge multiplicities).

A laminar family of nontrivial tight cuts $\mathcal{F}$ is said to satisfy the *odd cycle property* if every $\mathcal{F}$-contraction is nonbipartite. A nontrivial tight cut $\delta(U)$ has the *odd cycle property* if the family $\mathcal{F} = \{\delta(U)\}$ has the odd cycle property. For a laminar family $\mathcal{F}$ of nontrivial tight cuts and a nontrivial tight cut $\delta(U)$ that is laminar with respect to each cut in $\mathcal{F}$, we denote by $\mathcal{F}_{G/U}$ the set of cuts in $\mathcal{F}$ that remain nontrivial tight cuts in G/U.

Let $\mathfrak{F}_G$ denote the family whose elements are maximal inclusionwise laminar families of non-trivial tight cuts having the odd cycle property. Every $\mathcal{F} \in \mathfrak{F}_G$ has the same cardinality, which is the number of bricks of G minus one [35]. When $\mathfrak{F}_G = \{\varnothing\}$ and G is nonbipartite, G is called *near-brick*. Equivalently, a near-brick is a nonbipartite matching covered graph such that no tight cut has the odd cycle property. In particular, a near-brick has a single brick, and any brick is a near-brick.

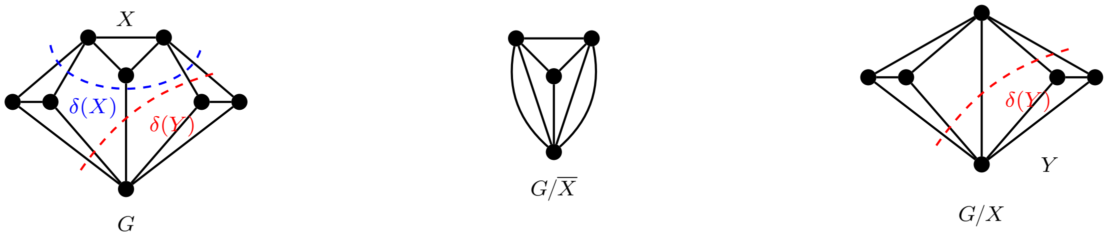


Fig. 1: A matching covered graph G, and a maximal incusionwise laminar family of tight cuts $\mathcal{F} = \{\delta(X), \delta(Y)\}$. As $\mathcal{F} \in \mathfrak{F}_G$, G has three bricks. In particular, one can see the two $\delta(X)$-contractions G/X and $G/\overline{X}$. While $G/\overline{X}$ is already one of the bricks of G, G/X still has two bricks that one can identify by computing $G/X/Y$ and $G/X/\overline{Y}$. All of these bricks are isomorphic to K_4, up to parallel edges.

In [17], the following connection between the solidity of a graph and the one of its bricks is established.

Theorem 8 (de Carvalho et al. [17]) A matching covered graph is solid if and only if all its bricks are.

By de Carvalho et al. [17, Lemma 2.29], odd intercyclic matching covered graphs are solid, such as bipartite matching covered graphs, odd wheels, and Möbius ladders of even order [19].

2.3 The matching polytope.

Given a graph $G = (V, E)$, the matching polytope of G is denoted by $P_{\mathrm{M}}(G)$. The following system of inequalities describes $P_{\mathrm{M}}(G)$ and is known as *Edmonds' system* [22].

$$(1) \begin{cases} x(E(U)) \leqslant (|U| - 1)/2, & \text{for each } U \subseteq V \text{ with } |U| \geqslant 3 \text{ odd}, \quad (1a) \\ x(\delta(u)) \leqslant 1, & \text{for each } u \in V, \quad (1b) \\ x \geqslant \mathbf{0}. & \quad (1c) \end{cases}$$

Cunningham and Marsh [15] proved that Edmonds' system is always TDI. In [20], the authors characterized the graphs for which Edmonds' system is box-TDI. A graph H is a *fully odd subdivision* of a graph G if H is obtained from G by subdividing each edge of G into a path of odd length (possibly the length is one), where the *length* is the number of edges in the path. Since Edmonds' system is always TDI [16], its box-TDIness is equivalent to that of the underlying polytope [12], hence we can restate their result as follows.

Theorem 9 (Ding et al. [20]) The matching polytope of a graph G is box-TDI if and only if G has no fully odd subdivision of G_1, G_2, G_3, and G_4 of Figure 2 as a subgraph.

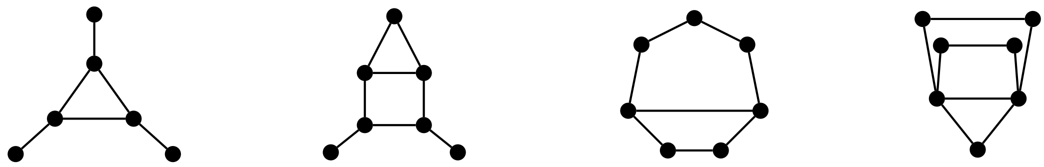


Fig. 2: The graphs G_i, $i = 1, 2, 3, 4$, are the subgraphs that prevent the matching polytope from being box-TDI (up to fully odd subdivision).

2.4 The perfect matching polytope.

The perfect matching polytope $P(G)$ of a graph G is described by the TDI system obtained from Edmonds' system by setting (1b) to equality. However, the following system also describes the perfect matching polytope [22], and is more convenient to investigate its box-TDIness:

$$(2)\begin{cases} x(\delta(U)) \geqslant 1, & \text{for each } U \subseteq V \text{ with } U| \geqslant 3 \text{ odd}, \quad (2a)\\ x(\delta(u)) = 1, & \text{for each } u \in V, \quad (2b)\\ x \geqslant \mathbf{0} & (2c)\end{cases}$$

Note that for bipartite graphs, inequalities (2a) are redundant, and the remaining system (2b)-(2c) is box-TDI by the total unimodularity of the incidence matrix [38, Section 19.3].

Let G be a matching covered graph and $\mathcal{F} \in \mathfrak{F}_G$. We define $M_G^{\mathcal{F}}$ as the matrix whose $|V|+|\mathcal{F}|$ rows are associated with the left-hand sides of equalities (2b) and inequalities (2a) associated with the cuts of $\mathcal{F}$. When $\mathfrak{F}_G = \{\varnothing\}$, $M_G^{\mathcal{F}}$ corresponds to the vertex-edge incidence matrix.

In their seminal works, Naddef [36], Edmonds et al. [23], and Lovász [35], proved that whenever G is nonbipartite, then $M_G^{\mathcal{F}}$ has full row rank. Moreover, the maximum number of linearly independent perfect matchings is $|E| - |V| - |\mathcal{F}| + 1$. We restate the results of Naddef [36], Edmonds et al. [23], and Lovász [35] in terms of face-defining matrices.

Theorem 10 ([23, 35, 36]) Let G be a nonbipartite matching covered graph. Then, $M_G^{\mathcal{F}}$ is face-defining for $\operatorname{aff}(P(G))$ for every $\mathcal{F} \in \mathfrak{F}_G$.

In particular, the equality associated with any nontrivial tight cut is a linear combination of equalities associated with the trivial tight cuts and the tight cuts of $\mathcal{F}$.

2.5 Box-TDIness: matchings VS perfect matchings.

The box-TDIness of the matching polytope implies that of the perfect matching polytope, since the latter is a face of the former. However, the converse does not hold. For instance, the perfect matching polytope of the graph G_1 in Figure 2 is box-TDI — containing only a single point — while its matching polytope is not, as shown by Theorem 9.

This phenomenon also occurs for matching covered graphs: we provide below four infinite families of graphs whose perfect matching polytope is box-TDI, but whose matching polytope is not, as they contain one of the forbidden structures of Theorem 9. Indeed, by Theorem 17 and Corollary 19, the perfect matching polytope of any fully odd subdivision of the graphs G_1', G_2', G_3', and G_4' of Figure 3 is box-TDI. Their matching polytope is not box-TDI, as each of them contains one of the forbidden subgraphs of Theorem 9.

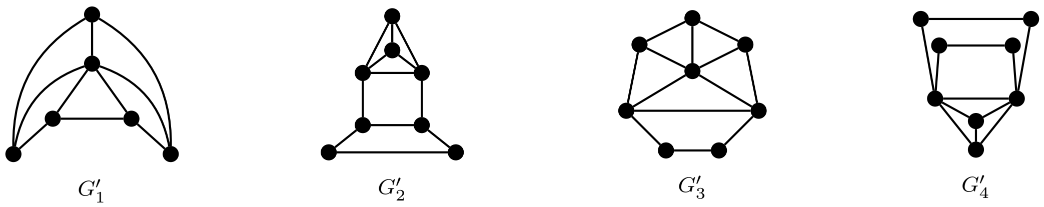


Fig. 3: The graphs G_1', G_2', G_3', and G_4' are matching covered graphs whose perfect matching polytope is box-TDI, yet their matching polytope is not.

3 When is the perfect matching polytope a slice of $\mathbb{R}^E_{\geqslant \mathbf{0}}$?

In [18, Theorem 2.1], de Carvalho et al. characterize the class of matching covered graphs for which the perfect matching polytope can be described by the system $x(\delta(u)) = 1$ for all $u \in V$, $x \geqslant \mathbf{0}$: they are the solid near-bricks and the bipartite graphs. In this section, following the line of their proof, we generalize this result and prove that the perfect matching polytope of a graph is a slice of nonnegative orthant if and only if the graph is solid. This yields a compact description for the perfect matching polytope of solid graphs, whereas this polytope has no compact formulation in general [37].

Theorem 11 Let $G = (V, E)$ be a matching covered graph. Then, $P(G) = \text{aff}(P(G)) \cap \mathbb{R}^E_{\geqslant 0}$ if and only if G is solid.

Proof We prove the result for nonbipartite graphs, as for bipartite graphs it holds thanks to the total unimodularity of the incidence matrix [30]. Let $\mathcal{F} \in \mathfrak{F}_G$. By Theorem 10, it is sufficient to prove that $P(G) = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$ if and only if G is solid.

($\Rightarrow$) Suppose that $G = (V, E)$ is not solid. Then, there exists a separating cut C which is not tight, and hence a perfect matching M of G such that $|M \cap C| > 1$. In [17, Lemma 2.19], de Carvalho et al. proved that a cut is separating if and only if for each edge of the cut, there exists a perfect matching that contains this edge and exactly one edge in the cut. Hence, since C is separating, for every edge $e \in M$, there exists a perfect matching M_e of G including e for which $|M_e \cap C| = 1$. Then, let $p = \frac{1}{|M|-1}((\sum_{e \in M} \chi^{M_e}) - \chi^M)$. By construction, $p \in \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$, since M_e, $e \in E$, and M intersect each tight cut exactly once by definition. But $p^\top \chi^C < 1$, and since C is separating, its shores have odd cardinality, and hence $p \notin P(G)$.

($\Leftarrow$) Let $P = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$, suppose that $P \neq P(G)$, and let us prove that G is not solid. Since $P(G) \subseteq P$ and every integer point of P is the characteristic vector of a perfect matching of G, P has a fractional vertex $p \notin P(G)$. Then, let $\mathcal{C}$ be the family of cuts associated with inequalities (2a) that are not satisfied by p. By Theorem 10, every point of P satisfies to equality the inequalities (2a) associated with tight cuts. Hence, no cut in $\mathcal{C}$ is tight. Let $\mathcal{M}$ denote the family of all perfect matchings of G. Let $\delta(U)$ be a cut in $\mathcal{C}$ such that there exists no cut $C \in \mathcal{C}$ for which $|C \cap M| \leqslant |\delta(U) \cap M|$ for all M in $\mathcal{M}$ and $|C \cap M'| < |\delta(U) \cap M'|$ for some $M' \in \mathcal{M}$.

Let us prove that $\delta(U)$ is separating, that is, G/U and $G/\overline{U}$ are both matching covered. This will contradict the solidity of G. Let u be the contraction of U in G/U and suppose that G/U is not matching covered. Then, by Theorems 5 and 6, there exists a node set S such that $|\mathcal{O}((G/U)\backslash S)| > |S|$ or S is a barrier with adjacent nodes. In both cases, u belongs to S, as otherwise G is not matching covered since $|U|$ is odd. Since $\bigcup_{K \in \mathcal{O}((G/U)\backslash S)} \delta(K)$ is contained in

$\delta(S)$, we have

$$\begin{aligned}\sum_{K\in\mathcal{O}((G/U)\backslash S)} p^\top\chi^{\delta(K)} &\leqslant p^\top\chi^{\delta(S)}\\ &= p^\top\chi^{\delta(u)\backslash E(S)} + \sum_{s\in S\backslash\{u\}} p^\top\chi^{\delta(s)\backslash E(S)}\\ &\leqslant p^\top\chi^{\delta(U)} + \sum_{s\in S\backslash\{u\}} p^\top\chi^{\delta(s)}\\ &< 1 + (|S|-1),\end{aligned}$$

where the last inequality holds since p violates the inequality (2a) associated with U but satisfies (2b). Thus, there is an odd component L of $(G/U)\backslash S$ such that $p^\top\chi^{\delta(L)} < 1$. Equalities (2b) imply that L is nontrivial so $L \in \mathcal{C}$. Note that every odd component of $(G/U)\backslash S$ is an odd component of $G\backslash(S\backslash\{u\}\cup U)$. For every perfect matching M of G, we have

$$\begin{aligned}|\delta(U)\cap M| + |\mathcal{O}((G/U)\backslash S)| - 1 &\geqslant |\delta(U)\cap M| + |S| - 1\\ &= |\delta(U)\cap M| + \sum_{s\in S\backslash\{u\}} |\delta(s)\cap M|\\ &\geqslant |\delta(S\backslash\{u\}\cup U)\cap M|\\ &\geqslant |(\bigcup_{K\in\mathcal{O}((G/U)\backslash S)} \delta(K))\cap M|\\ &= \sum_{K\in\mathcal{O}((G/U)\backslash S)} |\delta(K)\cap M|\\ &\geqslant |\delta(L)\cap M| + |\mathcal{O}((G/U)\backslash S)| - 1,\end{aligned}$$

where the last inequality holds because $|\delta(K)\cap M| \geqslant 1$ for all $K \in \mathcal{O}((G/U)\backslash S)$. This implies that $|\delta(L)\cap M| \leqslant |\delta(U)\cap M|$ for every perfect matching M of G.

Let us show that this is impossible. First, if $|\mathcal{O}((G/U)\backslash S)| > |S|$, then:

$$\begin{aligned}|\mathcal{O}((G/U)\backslash S)| - 1 + |M\cap\delta(U)| &> |S| - 1 + |M\cap\delta(U)|\\ &\geqslant |M\cap\delta(\mathcal{O}((G/U)\backslash S))|\\ &\geqslant |\mathcal{O}((G/U)\backslash S)| - 1 + |M\cap\delta(L)|,\end{aligned}$$

which contradicts the choice of $\delta(U)$ in $\mathcal{C}$.

Otherwise, we have $|\mathcal{O}((G/U)\backslash S)| = |S|$ and, by Theorem 6, there exists an edge $e \in S$, and since G is matching covered, there exists a perfect matching M_e of G including e. Then, we have:

$$\begin{aligned}|\mathcal{O}((G/U)\backslash S)| + |M_e\cap\delta(U)| &= |S| + |M_e\cap\delta(U)|\\ &\geqslant |M_e\cap\delta(S)| + 2 + |M_e\cap\delta(U)|\\ &\geqslant |M_e\cap\delta(\mathcal{O}((G/U)\backslash S))| + 2\\ &\geqslant |\mathcal{O}((G/U)\backslash S)| + |M_e\cap\delta(L)| + 2,\end{aligned}$$

and M_e contradicts the choice of $\delta(U)$ in $\mathcal{C}$.

Therefore, G/U is matching covered. Similarly, $G/\overline{U}$ is matching covered. Hence, $\delta(U)$ is separating and G is not solid. □

Moreover, it turns out that the box-TDIness of a perfect matching polytope is captured by that of its affine hull.

Theorem 12 Let G be a nonbipartite matching covered graph and $\mathcal{F} \in \mathfrak{F}_G$. Then, the following statements are equivalent:

1. $P(G)$ is box-TDI;
2. aff$(P(G))$ is box-TDI;
3. $M_G^{\mathcal{F}}$ is equimodular;
4. $P(G) = \{x\colon Mx = b, x \geqslant \mathbf{0}\}$ with M totally unimodular.

Moreover, if $P(G)$ is box-TDI, then $P(G) = \text{aff}(P(G)) \cap \mathbb{R}^E_{\geqslant 0}$.

Proof (1. $\Rightarrow$ 2.) Every face of a box-TDI polyhedron is also box-TDI [24, (10.13)].

(2. $\Leftrightarrow$ 3.) Since an affine space has a single face, this follows from Theorems 3 and 10.

(2.&3. $\Rightarrow$ 4.) By Theorem 10, aff$(P(G)) = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}\}$. Since aff$(P(G))$ is integer and box-TDI, so is aff$(P(G)) \cap \mathbb{R}^E_{\geqslant 0}$. In particular, the latter is $P(G)$. Let B be a basis of $M_G^{\mathcal{F}}$, $M = B^{-1} M_G^{\mathcal{F}}$, and $b = B^{-1}\mathbf{1}$. Since $M_G^{\mathcal{F}}$ is equimodular, M is totally unimodular by Theorem 1, and $P(G) = \{x\colon Mx = b, x \geqslant \mathbf{0}\}$. Note that b is integer since $\mathbf{1}$ is an integer combination of the columns of $M_G^{\mathcal{F}}$.

(4. $\Rightarrow$ 1.) When M is totally unimodular, $\{x\colon Mx = b, x \geqslant \mathbf{0}\}$ is box-TDI [30]. □

By Theorem 12, when $P(G)$ is box-TDI, it is the intersection of its affine hull and the nonnegative orthant. Moreover, in Statement 4, M has full row rank and is totally unimodular, so the associated system describing $P(G)$ is a *Schriver's system* [13], that is, it is a minimal TDI system describing $P(G)$.

As a consequence of Theorem 12, the box-TDIness of perfect matching polytopes can be checked in polynomial time. First, given a matching covered graph G, one can obtain a family $\mathcal{F} \in \mathfrak{F}_G$ by applying the brick decomposition algorithm of Edmonds et al. [23] and build the matrix $M_G^{\mathcal{F}}$ in polynomial time. By Statement 3 of Theorem 12, the perfect matching polytope of G is box-TDI if and only if $M_G^{\mathcal{F}}$ is equimodular. By Theorem 1, the equimodularity of the latter can be checked as follows: find a basis B of $M_G^{\mathcal{F}}$, and then test the total unimodularity of $B^{-1} M_G^{\mathcal{F}}$. This can be done in polynomial time [41]. This polynomial case stands in contrast to the general case, where determining whether a given polytope is box-TDI is co-NP-complete [10].

Corollary 13 Deciding whether the perfect matching polytope of a matching covered graph is box-TDI can be done in polynomial time.

4 Near-bricks and bicritical graphs

In this section, we characterize the box-TDIness of the perfect matching polytope of near-bricks using forbidden subgraphs. We also characterize the box-TDIness of perfect matching polytopes of bicritical graphs through tight cut decomposition. As a consequence of these results, we end the section by providing two specific graphs: the first highlights that a natural necessary condition for a perfect matching polytope to be box-TDI is insufficient, and the second that a characterization of box-TDI perfect matching polytopes in terms of forbidden subgraphs is impossible.

We first prove that tight cut contractions preserve the box-TDIness of the perfect matching polytope.

Lemma 14 Let G be a matching covered graph. If $P(G)$ is box-TDI, then so is $P(G/U)$ for each tight cut $\delta(U)$.

Proof Since $\delta(U)$ is tight and G is matching covered, $P(G/U)$ is the orthogonal projection of $P(G)$ onto the coordinates indexed by $\delta(U) \cup E(\overline{U})$. Such projections preserve box-TDIness [38, Page 323], and the result follows. □

4.1 The case of near-bricks.

Recall that a near-brick is a nonbipartite matching covered graph in which no tight cut has the odd cycle property.

Lemma 15 Let $G = (V, E)$ be a matching covered graph and $\delta(X)$ be a tight cut of G without the odd cycle property. If G is odd-intercyclic, so is G/X.

Proof Suppose that G/X contains two disjoint odd cycles C_1 and C_2. If C_1 and C_2 are cycles of G, then G is not odd intercyclic and the proof ends. Otherwise, one of the two cycles contains the vertex u obtained from the contraction of X. Without loss of generality, suppose $u \in V(C_1)$. Since $\delta(X)$ does not have the odd cycle property and G/X is non-bipartite, $G/\overline{X}$ is bipartite. Moreover, it is matching covered since $\delta(X)$ is tight, and hence it is 2-connected. Then, there exists an even cycle C_3 of $G/\overline{X}$ containing u. In particular, $(C_1 \backslash \delta(u)) \cup C_3$ is an odd cycle of G disjoint from C_2, which ends the proof. □

Since the brick of a near-brick is obtained by contracting tight cuts without the odd cycle property, Lemma 15 implies the following.

Corollary 16 If a near-brick is odd intercyclic, then so is its brick.

As a consequence, we characterize which near-bricks have a box-TDI perfect matching polytope.

Theorem 17 The perfect matching polytope of a near-brick is box-TDI if and only if the near-brick is odd intercyclic.

Proof Let $G = (V, E)$ be a near-brick.

($\Rightarrow$) We equivalently prove that if G has two vertex-disjoint odd circuits, then $P(G)$ is not box-TDI. Suppose that G contains two vertex-disjoint odd circuits. Since G is a near-brick, $\mathfrak{F}_G = \{\varnothing\}$. Thus, A_G is face-defining for aff($P(G)$), by Theorem 10. By Theorem 2, A_G is not equimodular. Thus, $P(G)$ is not box-TDI by Theorem 3.

($\Leftarrow$) Suppose that G is odd intercyclic. By Corollary 16, its brick is odd intercyclic, hence, is a solid graph by [17, Lemma 2.29]. Thus, by Theorem 11, $P(G) = \text{aff}(P(G)) \cap \{x \geqslant \mathbf{0}\}$. By Theorem 2 and Theorem 10, A_G is an equimodular face-defining matrix for aff($P(G)$). Thus, $P(G) = \{x \colon A_G x = \mathbf{1}\} \cap \{x \colon x \geqslant \mathbf{0}\}$ is box-TDI by Theorem 3 and the definition of box-TDIness. □

4.2 The case of bicritical graphs.

Let u and w be two vertices of a matching covered graph G such that $G \backslash \{u, w\}$ has precisely two connected components with vertex sets U and W of even size. Then, $\delta(X)$, with $X = U \cup \{u\}$, is a 2-*separation cut with respect to* u and w. Similarly, $\delta(W \cup \{u\})$ is also a 2-separation cut with respect to u and w. An illustration of 2-separation cuts is given in Figure 4. Note that every 2-separation cut is tight.

It turns out that the converse of Lemma 14 holds for 2-separation cuts.

Theorem 18 Let $\delta(X)$ be a 2-separation cut of a nonbipartite matching covered graph G. Then, $P(G)$ is box-TDI if and only if both $P(G/X)$ and $P(G/\overline{X})$ are.

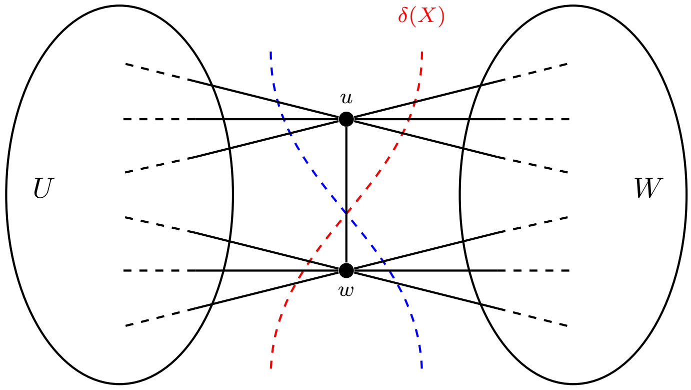


Fig. 4: The two 2-separation cuts with respect to u and w are drawn in red and blue. Note that u and w may be nonadjacent.

Proof In this proof, we use the following notation for a matrix A whose columns are indexed by a subset of edges of a graph: $E(A)$ denotes this set of edges.

The "only if" part comes from Lemma 14, hence let us prove the "if" part.

Let u and w be two vertices such that $G \backslash \{u, w\}$ has precisely two even connected components $G[U]$ and $G[W]$. Let $X = U \cup \{u\}$. By definition, $\delta(X)$ is a 2-separation cut with respect to u and w. Suppose that $P(G/X)$ and $P(G/\overline{X})$ are box-TDI and let us show that $P(G)$ is.

We can restrict the proof to the case where $uw \in E(G)$. Indeed, suppose the contrary. Since G is matching covered and $\delta(X)$ is tight, both G/X and $G/\overline{X}$ contain the edge uw. Adding or removing edge duplicates preserves box-TDIness, so the box-TDIness of $P(G/X)$ and $P(G/\overline{X})$ implies the one of $P((G/X)+uw)$ and $P((G/\overline{X})+uw)$. Note that $(G/X)+uw$ and $(G/\overline{X})+uw$ are the $\delta(X)$-contractions of $G+uw$. Since G is matching covered so is $G+uw$. Then, proving the box-TDIness of $P(G)$ from the one of $P(G/X)$ and $P(G/\overline{X})$ can be done by proving the box-TDIness of $P(G+uw)$ from the one of $P((G/X)+uw)$ and $P((G/\overline{X})+uw)$ as $P(G)$ is the projection onto $\mathbb{R}^{E(G)}$ of the intersection of $P(G+uw)$ with the box $\{x \in \mathbb{R}^{E(G+uw)} : x_{uw} = 0\}$. Hence, for the rest of the proof, we suppose that $uw \in E(G)$.

Consider a family $\mathcal{F} \in \mathfrak{F}_G$ such that $\mathcal{F}$ contains $\delta(X)$ if $\delta(X)$ has the odd cycle property, or such that $\mathcal{F} \cup \{\delta(X)\}$ is laminar otherwise. In both cases, the laminarity implies that the shores of the tight cuts of $\mathcal{F}$ are either contained in X or contain X. The cuts of former type correspond to rows $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$, the latter do not.

Let e be any edge of G. Up to replacing $U \cup \{u\}$ by $W \cup \{w\}$, we may assume that e belongs to $E(G[U \cup \{u, w\}])$. Let B be any basis of $M_G^{\mathcal{F}}$ not containing e, and denote by μ^e the e-th column of $M_G^{\mathcal{F}}$. Let us prove that there exists a 0,± 1 vector x such that $Bx = \mu^e$. By Theorem 1, this will imply the equimodularity of $M_G^{\mathcal{F}}$, and hence the box-TDIness of $P(G)$ by Statement 3 of Theorem 12.

Let K and H be the graphs respectively obtained from G/X and $G/\overline{X}$ by removing edges parallel to uw. Then, e is in $E(H)$, and $P(H)$ is box-TDI as the removal of duplicate edges does not impact box-TDIness. Let B_H (resp. B_K) be the submatrix of B whose rows are indexed by $\delta(Y)$ for $Y \subseteq X$ (resp. $Y \supseteq X$) and whose columns are indexed by the edges of H (resp. K) corresponding to columns of B. Let ν^e denote the e-th column of $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$. Note that ν^e is the restriction of μ^e to the rows of $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$.

Our proof is in two steps: we first prove that there exist a basis C_H of B_H and a vector $y \in \{0, \pm 1\}^{E(C_H)}$ such that $C_H y = \nu^e$. We then construct from y a vector $x \in \{0, \pm 1\}^{E(B)}$ such that $Bx = \mu^e$.

Step 1. First, suppose that H is bipartite. Then, $\delta(X) \notin \mathcal{F}$ and no cut of $\mathcal{F}$ has a shore contained in X. In particular, note that $[B_H, \nu^e]$ is a submatrix of A_H, and the rows of B_H are indexed by $\delta(h)$ for all $h \in V(H) \backslash \{w\}$. Since H is bipartite and connected, any matrix obtained from A_H by removing a single row, like B_H, has full row rank and is totally unimodular. Let C_H be a basis of B_H. Then, there exists a 0,$\pm$1 vector y such that $C_H y = \nu^e$.

Suppose now that H is nonbipartite. Let M be the matrix obtained from $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$ by removing the duplicates of column uw. Then, B_H is the submatrix of M whose columns are indexed by those of B contained in M. Hence, a basis C_H of B_H is also a basis of M. By Theorem 10, M is face-defining for $P(H)$. Observe that M is a submatrix of $M_G^{\mathcal{F}}$. By Theorem 1 and by Statement 3 of Theorem 12, since $P(H)$ is box-TDI, there exists a 0,$\pm$1 vector y such that $C_H y = \nu^e$ for some basis C_H of B_H.

Without loss of generality, we suppose that in both cases, we choose a basis C_H such that $uw \notin E(C_H)$ if such a basis exists.

Step 2. There are two cases. First, suppose that either $uw \notin E(C_H)$ or $y_{uw} = 0$. Then, every non-null coordinate of y is associated with an edge that does not belong to G/X. Hence the associated column in B_K only contains 0. Moreover, the restriction of μ_e to the rows of B_K is the 0 vector. Defining $x \in \{0, \pm 1\}^{E(B)}$ such that $x_e = y_e$ for all $e \in E(C_H)$, and $x_e = 0$ for $e \in E(B) \backslash E(C_H)$ gives $Bx = \mu^e$.

The second case is when $uw \in E(C_H)$ and $y_{uw} \neq 0$. Then, $e \neq uw$ and every basis of B_H contains uw, from our choice of C_H and $uw \in E(C_H)$.

Consequently, there exists a basis of B_K not containing uw. Indeed, suppose by contradiction that every basis of B_K contains the edge uw. Then, removing column uw in B_K gives a matrix with rank at most $|V(K)| + |\mathcal{F}_K| - 1$. Similarly, from our choice of C_H, since $uw \in E(C_H)$, then every basis of B_H contains uw, so removing uw from B_H gives a matrix with rank at most $|V(H)| + |\mathcal{F}_H| - 1$. Hence, the rank of B without uw is at most $|V(G)| + |\mathcal{F}| - 2$, which contradicts that B has full row rank and $|V(G)| + |\mathcal{F}|$ rows.

Without loss of generality, let C_K be a basis of B_K not containing uw. Since $P(K)$ is box-TDI, there exists a 0,$\pm$1 vector z such that $C_K z = \rho^{uw}$, where ρ^{uw} is the restriction of the column of $M_G^{\mathcal{F}}$ associated with uw to the rows of B_K. Now, define the vector $\hat{x} \in \{0,\pm 1\}^E$ as follows:

$$\hat{x}_f = \begin{cases} y_f & \text{if } f \in E(C_H) \backslash \{uw\}, \\ y_{uw} z_f & \text{if } f \in E(C_K), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for $F \subseteq E(G)$, since $\hat{x}_{uw} = 0$ and since uw is in $E(C_H)$, we have:

$$\hat{x}(F) = \begin{cases} y(F \cap E(C_H)) + y_{uw} z(F \cap E(C_K)) & \text{if } uw \notin F, \\ y(F \cap E(C_H)) - y_{uw} + y_{uw} z(F \cap E(C_K)) & \text{if } uw \in F. \end{cases} \tag{3}$$

We end the proof by showing that for each tight cut $\delta(Y)$ indexing a row of B, we have $\hat{x}(\delta(Y)) = \mu^e_{\delta(Y)}$. This immediately implies that the restriction x of $\hat{x}$ to the coordinates indexed by the columns of B satisfies $Bx = \mu^e$, since the only nonzero coordinates of $\hat{x}$ are associated with columns of $E(B)$. Let $\delta(Y)$ be a tight cut indexing a row of B.

Suppose first that $uw \notin \delta(Y)$. Then, either $Y \subseteq U$ or $Y \subseteq W$. In the former case, $\delta(Y) \cap E(C_K) = \varnothing$ so by (3), $\hat{x}(\delta(Y)) = y(\delta(Y) \cap E(C_H)) = \mu^e_{\delta(Y)}$ since $C_H y = \nu^e$. In the latter case, $\delta(Y) \cap E(C_H) = \varnothing$ so, by (3), $\hat{x}(\delta(Y)) = y_{uw} z(\delta(Y) \cap E(C_K)) = 0$ since $C_K z = \rho^{uw}$ and $uw \notin \delta(Y)$. Moreover, $\mu^e_{\delta(Y)} = 0$ since e is in G/X, hence not in $\delta(Y)$.

Suppose now that $uw \in \delta(Y)$. Then, we have $\rho^{uw}_{\delta(Y)} = 1$. By first applying (3), then $y(\delta(Y) \cap E(C_H)) = \mu^e_{\delta(Y)}$, and finally $z(\delta(Y) \cap E(C_K)) = \rho^{uw}_{\delta(Y)} = 1$, we obtain:

$$\hat{x}(\delta(Y)) = y(\delta(Y) \cap E(C_H)) - y_{uw} + y_{uw} z(\delta(Y) \cap E(C_K)) = \mu^e_{\delta(Y)}.$$

□

Fully odd subdividing a graph preserves the nonbox-TDIness of its matching polytope, yet the converse does not hold [20]. For perfect matchings, it is an equivalence which follows from Theorem 18.

Corollary 19 The perfect matching polytope of a fully odd subdivision of a matching covered graph G is box-TDI if and only if $P(G)$ is.

Proof Let H be the graph obtained by replacing the edge uv of G with the path $u, u_1, \ldots, u_{2k}, v$, with $k \in \mathbb{Z}_{>0}$. Note that $\delta(\{u, u_1, \ldots, u_{2k}\})$ is a 2-separation cut of H, and the $\delta(\{u, u_1, \ldots, u_{2k}\})$-contractions are G and C_{2k+2}. The latter is bipartite, hence its perfect matching polytope is box-TDI. Therefore, by Theorem 18, $P(H)$ is box-TDI if and only if $P(G)$ is. □

A graph $G = (V, E)$ is *bicritical* if $G \backslash \{u, v\}$ has a perfect matching for every $u, v \in V$. Edmonds et al. [23, Section 2, page 252] — referring to $\{u, v\}$ as an articulation set — proved that a bicritical graph is a matching covered graph whose tight cut decomposition can be accomplished by a sequence of tight cut contractions stepping exclusively in 2-separation cuts. Then, Theorems 17 and 18 and Corollary 19 give the following.

Corollary 20 The perfect matching polytope of a fully odd subdivision of a bicritical graph is box-TDI if and only if all the bricks of the graph are odd intercyclic.

An *odd wheel* is a graph composed of a circuit of odd length, and an additional vertex connected to all the vertices of the circuit. In [17], the authors proved that the only solid planar bricks are odd wheels. Thus, by Corollary 19, Theorem 8, and [17, Lemma 2.29], the following holds.

Corollary 21 The perfect matching polytope of a fully odd subdivision of a bicritical planar graph is box-TDI if and only if all the bricks of the graph are odd wheels.

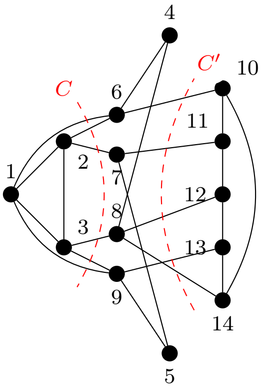
4.3 A necessary unsufficient condition.

Lemma 14 and Theorem 17 immediately give the following.

Corollary 22 Let G be a matching covered graph and $\mathcal{F} \in \mathfrak{F}_G$. If $P(G)$ is box-TDI, then the near-bricks obtained by any sequence contracting all the cuts of $\mathcal{F}$ are odd intercyclic.

Proof By Lemma 14, if $P(G)$ is box-TDI, then so is the perfect matching polytope of any near-brick that arises from a sequence of contractions of tight cuts of G. By Theorem 17, all these near-bricks are odd intercyclic. □

The converse of Corollary 22 does not hold. Figure 5 provides the Moonfish graph $\mathcal{M}$, which is a graph with two bricks illustrating this. Specifically, $\mathfrak{F}_{\mathcal{M}} = \{\{C\}, \{C'\}\}$, and all C and C'-contractions are odd intercyclic. However, by Theorems 3 and 10, $P(\mathcal{M})$ is not box-TDI, as $\{C\} \in \mathfrak{F}_{\mathcal{M}}$ and $M^{\{C\}}_{\mathcal{M}}$ is not equimodular.



	(1,2)	(1,3)	(1,6)	(1,9)	(2,3)	(2,7)	(3,8)	(4,6)	(5,7)	(6,10)	(7,11)	(8,12)	(8,14)	(9,13)	(10,11)
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0
3	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
6	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
7	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0
9	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
C	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0

	(1,2)	(1,6)	(2,6)	(2,7)	(3,8)	(3,9)	(4,6)	(5,7)	(6,10)	(7,11)	(8,12)	(8,14)	(9,13)	(10,11)	(12,13)
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
6	0	1	1	0	0	0	1	0	1	0	0	0	0	0	0
7	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0
8	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0
9	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
10	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
11	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
12	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
14	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
C	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0

Fig. 5: The Moonfish graph $\mathcal{M}$ and the unique maximal inclusionwise laminar family $\{C, C'\}$ of $\mathcal{M}$. One can check that all C-contractions and C'-contractions are odd intercyclic near-bricks. Since, $\mathcal{M}$ has two bricks —K_4 and the odd wheel of order 5— and the two C-contractions are nonbipartite, $\{C\} \in \mathfrak{F}_{\mathcal{M}}$. In particular, the two matrices above are nonsingular maximal submatrices of on the $M_{\mathcal{M}}^{\{C\}}$ and have —from left to right— determinant 4 and 8, up to the sign. Then, the matrix $M_{\mathcal{M}}^{\{C\}}$ is nonequimodular, by definition, and it is face-defining for the affine hull of $P(\mathcal{M})$ by Theorem 10. By Theorem 12, $P(\mathcal{M})$ is not box-TDI.

4.4 An impossible forbidden subgraph characterization.

In this section, we provide a graph whose perfect matching polytope is box-TDI, and yet contains a subgraph whose perfect matching polytope is not box-TDI. This highlights that, unlike the elegant characterization of box-TDI matching polytopes given in Theorem 9, it is not possible to characterize box-TDI perfect matching polytopes solely in terms of forbidden subgraphs. This limitation arises because, in the context of perfect matchings, using a subgraph approach requires that if an edge is deleted, all edges that appear exclusively in perfect matchings containing the deleted edge must also be removed.

Let G be the graph illustrated in Figure 6. By Theorem 8 and [17, Lemma 2.29] G is solid, since the bricks obtained with the tight cut decomposition associated with the family

$$\mathcal{H} = \{\delta(\{1,2,3\}, \delta(\{5,6,7\}, \delta(\{8,9,10\}), \delta(\{11,12,13\}), \delta(\{14,15,16\})\}$$

are all K_4. A maximal laminar subfamily of nontrivial tight cuts with the odd cycle property of $\mathcal{H}$ is

$$\mathcal{F} = \{\delta(\{1,2,3\}, \delta(\{5,6,7\}, \delta(\{8,9,10\}), \delta(\{11,12,13\})\}.$$

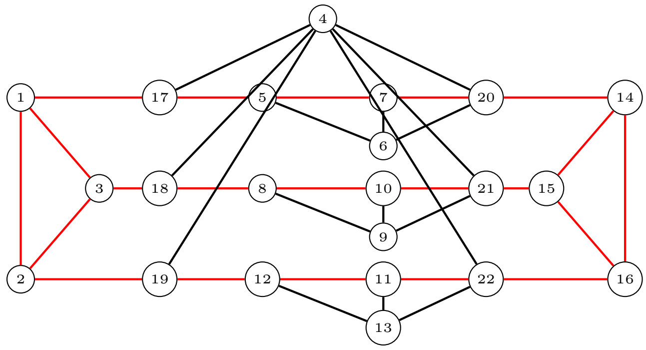


Fig. 6: A matching covered graph containing a fully odd subdivision of $\overline{C}_6$ (in red) as a subgraph and whose perfect matching polytope is box-TDI.

The family $F \in \mathfrak{F}_G$ gives the matrix $M_G^{\mathcal{F}}$ which is equimodular[1]. By Theorem 12, $P(G)$ is box-TDI.

The perfect matching polytope of the brick $\overline{C}_6$ is not box-TDI by Theorem 17. Hence, none is any of its fully odd subdivision by Corollary 19. Since the subgraph in red in Figure 6 is a fully odd subdivision of $\overline{C}_6$, it implies that G contains a subgraph whose perfect matching polytope is not box-TDI whereas the perfect matching polytope of G is.

5 Integer decomposition property

As a consequence of Theorems 12 and 4, box-TDI perfect matching polytopes have the integer decomposition property. In this section, we further discuss this property and its connection with d-edge-colorability.

The perfect matching *lattice* of a graph G is the set of integer combinations of incidence vectors of perfect matchings of G. Lovász [34] proved that a vector belongs to the perfect matching lattice of a matching covered graph G if and only if all its restrictions to the bricks of G belong to their perfect matching lattices. We observe that a similar result holds regarding the integer decomposition property. By projection, it is immediate that contracting a tight cut preserves the integer decomposition property of the perfect matching polytope. This is turned into an equivalence as follows.

Lemma 23 Let G be a matching covered graph and $\delta(U)$ a tight cut of G. Then, $P(G)$ has the integer decomposition property if and only if both $P(G/U)$ and $P(G/\overline{U})$ have it.

Proof Let $k \in \mathbb{Z}_{\geqslant 0}$ and $x \in kP(G)$. Denote by $x_{|G/U}$ the restriction of x to G/U. Since $\delta(U)$ is tight, note that $x_{|G/U} \in kP(G/U)$ and $x_{|G/\overline{U}} \in kP(G/\overline{U})$. Then, if both $P(G/U)$ and $P(G/\overline{U})$ have the integer decomposition property, $x_{|G/U}$ and $x_{|G/\overline{U}}$ are respectively the sum of k perfect matchings of G/U and $G/\overline{U}$. Since the contributions of $x_{|G/U}$ and $x_{|G/\overline{U}}$ to the edges of $\delta(U) = \delta(\overline{U})$ are identical, pairing appropriately these matchings of G/U and $G/\overline{U}$ decomposes x into k perfect matchings of G. ◻

Applied to a tight cut decomposition, Lemma 23 gives the the following.

[1] The equimodularity has been checked by enumerating all the nonzero maximal minors and verifying that they equal up to absolute value.

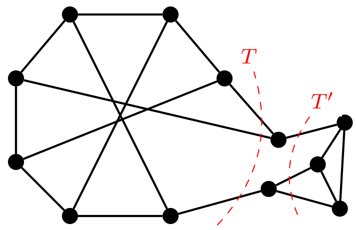


Fig. 7: A nonplanar nonbipartite cubic graph with two nontrivial tight cuts T and T', which is 3-edge-colorable by Corollary 26.

Corollary 24 The perfect matching polytope of a matching covered graph has the integer decomposition property if and only if the perfect matching polytope of each of its bricks has this property.

In [33, Section 2.1], the authors mention that they are not aware whether $0, 1$ polytopes with the integer decomposition property are all box-TDI. This is not the case, and actually it is the converse that holds for perfect matchings. Indeed, Theorem 4 and Statement 4 of Theorem 12 yield that box-TDI perfect matching polytopes have the integer decomposition property. However, $P(\overline{C}_6)$ has this property but is not box-TDI by Theorem 17. We provide a wider class of graphs whose perfect matching polytope has the integer decomposition property. The following is a consequence of Theorems 4 and 17 and Lemma 23.

Corollary 25 If all the bricks of a matching covered graph are odd intercylic, then its perfect matching polytope has the integer decomposition property.

In particular, the perfect matching polytope of the Moonfish graph in Figure 5 has the integer decomposition property.

A graph $G = (V, E)$ is *d-regular* if all its vertices have degree d, and it is a *d-graph* if in addition $|\delta(X)| \geqslant d$, for every $X \subseteq V$ with $|X|$ odd. Note that if G is a d-graph, deciding whether $\mathbf{1}$ belongs to the integer cone of the perfect matchings of G is equivalent to deciding whether G is d-edge-colorable. A d-edge-colorable d-graph is matching covered. Moreover, observe that tight cuts in a d-edge-colorable d-graph are all of size d, and that contracting such a cut yields a d-edge-colorable d-graph. Thus, a d-graph is d-edge-colorable if and only if all its bricks are.

Lemma 23 implies that if G is a d-graph and the perfect matching polytope of the bricks of G have IDP, then $P(G)$ has IDP, and hence G is d-edge-colorable. Deciding whether a d-regular graph is d-edge-colorable is NP-complete [31], for every nonnegative integer $d \geqslant 3$. The generalized Berge-Fulkerson conjecture, due to Seymour [42], states that if G is a d-graph, then there exist $2d$ perfect matchings of G covering twice every edge of G. Corollary 25 implies the following positive new case of the conjecture.

Corollary 26 Let G be a matching covered d-graph. If the bricks of G are odd intercyclic, then G is d-edge-colorable.

The four-color theorem is equivalent to the following: a 3-regular planar graph is 3-edge-colorable if and only if it is a 3-graph. Corollary 26 yields 3-edge-colorable 3-graphs which are non-necessarily planar graphs: for instance, Figure 7 provides a non-planar 3-edge-colorable whose bricks are K_4's or Möbius ladders of even order.

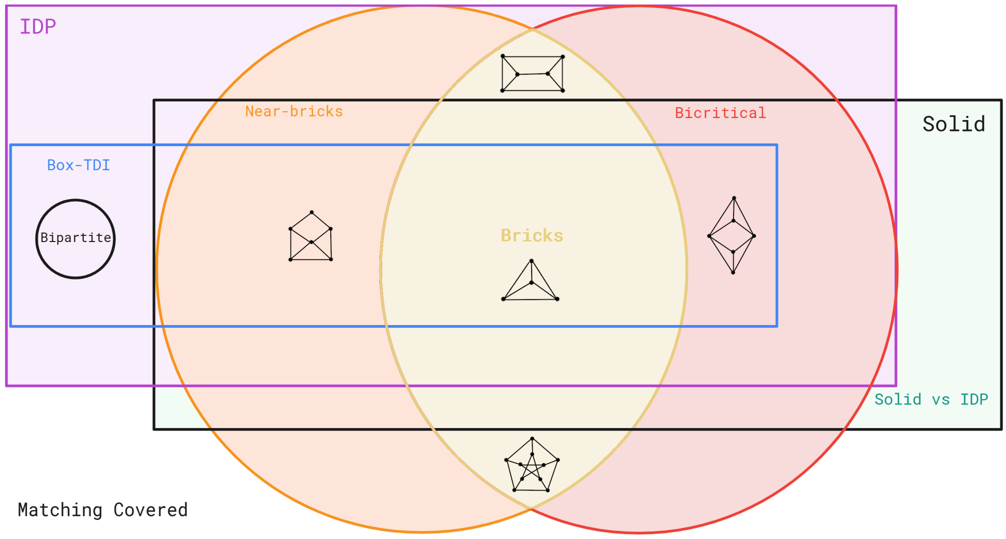


Fig. 8: The scheme of the classes of graphs studied in this paper. In yellow the bricks, which are a particular case of bicritical graphs (light red) and near-bricks (orange). Bipartite graphs, which by definition are neither near-brick nor bicritical graphs, are a special class of graphs whose perfect matching polytope is box-TDI. In light green solid graphs, while in light purple the graphs whose perfect matching polytope has IDP. Note that we currently do not know if the class of solid graphs is contained in IDP class. Among other examples, one can see depicted the well-known Petersen graph $\mathcal{P}$, which is the smallest non-3-edge-colorable 3-graph, and hence $P(\mathcal{P})$ does not have IDP.

Further questions

This paper explores integer properties of perfect matching polytopes and raises several further questions suggested by our findings. Figure 8 represents a panorama of the analyzed class of graphs in this work.

Concerning the box-TDIness of perfect matching polytopes, the impact of barrier cuts remains to be investigated, as Edmonds et al. [23] essentially proves that every matching covered graph admits a tight cut decomposition consisting only of 2-separation cuts and barrier cuts. In this paper, we have dealt with 2-separation cuts. Thus, answering the following question would finish the characterization of the box-TDIness of the perfect matching polytope in terms of tight cut contractions: *Can one characterize which barrier cut contractions preserve the box-TDIness of perfect matching polytopes?*

Additionally, Statement 4 of Theorem 12 prompts the following question, suggesting a potential min-max theorem: *Can an explicit totally unimodular matrix be found to describe box-TDI perfect matching polytopes?*

Lastly, our findings provide a new characterization of solid graphs, though it remains unknown whether an efficient recognition algorithm or graphic characterization exists for nonplanar solid graphs [6]: *Could this new polyhedral characterization help?*

Conflict of interest

The authors declare that they have no conflicts of interest.

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